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STUDYING OF OPTIMAL CONTROL PROBLEMS WITH NON-LOCAL INTEGRO-DIFFERENTIAL EQUATIONS

Speciality: 1211.01 – Differential equations

Field of science: Mathematics
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ABSTRACT

of the dissertation for the degree of Doctor of Philosophy

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The work was performed at Ganja State University the department of "Mathematical Analysis".

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GENERAL CHARACTERISTICS OF THE WORK

Rationale of the theme and development degree. Mathematical models of a number of problems in natural science are described by differential and integro-differential equations. There are processes that cannot be directly measured by the parameters that characterize them and however average value of main parameters of equation about these parameters is known at some points of region or on region itself or at the region where differential equation is defined or at some part of region. These problems are described by differential equations with nonlocal conditions. These problems are given in detailed at works of A.M. Nakhushev

1. Нахушев А.М. Задачи со смещением для уравнений в частных производных. М.: Наука, 2006. 287с.
2. Нахушев А.М. Уравнения математической биологии. М.: Высшая школа, 1995, 305с

and is indicated the specific areas in which these problems arose.

Academic A.A. Samarski

3. Самарский А.А. О некоторых проблемах теории дифференциальных уравнений // Дифференц.уравнения, 1980, т.16, №11, с.1925-1935

in his article he noted the need to study many problems in atomic physics, where nonlocal conditional problems arise.Since nonlocal boundary value problems occur in different fields of mechanics and physics, it is important to state and investigate of optimal control problems. Note that,

4. Cannon I.R., The solution of the heat equation subject to the specification of energy // Quart.Appl.Math., 1963, V.21, No.2, pp.155-160.
5. Carleman T. Sur la theorie des equations integrals et ses applications// Verhandlungen des Internat. Math. Kongr. Zurich. 1932.-1.-p. 138-151.

at their works, it is first time, the problems with nonlocal condition have studied.

In modern times, both boundary value problems with nonlocal condition and also optimal control problems are explored actively.In recent years, nonlocal boundary value problems by means of differential and integro-differential equations have been studied by M. Mardanov, K. Ayda-zade, V. Abdullayev, Y. Sharifov, K. Ismayilova and others have been researched in their works. Optimal control problems described in such problems are considered at the works of M. Mardanov, K. Mansimov, K. Ayda-zade, V. Abdullayev, Y. Sharifov.

The goal and tasks of the research. The main purpose of the dissertation is to study the system of differential and integro-differential equations with non-local condition and impulse effect and to study the optimal control problems described by them.The goal of the dissertation is to identify sufficient conditions for the existence and uniqueness of solution of the system of differential and integro-differential equations with non-local condition and impulse effect and the necessary conditions for optimization

Investigation methods. Differential equations and methods of optimal control theory were used in the dissertation.

The main thesis to be defined.

- Sufficient conditions have been found for the existence and uniqueness of the solution of non-local boundary value problems for a system of nonlinear differential equations with impulse effect.
- Sufficient conditions have been found for the existence and uniqueness of the solution of non-local boundary value problems for a system of nonlinear integro-differential equations with impulse effect.
- The continuous dependence of systems of nonlinear differential and integro-differential equations with impulse effect on the boundary conditions of problem have been studied.
- Necessary conditions for optimization in the form of variational inequality and Potryagin's maximum principle have been found in optimal control problems described by systems of differential

and integro-differential equations with nonlocal condition and impulse effect.

Scientific novelty of the research The dissertation work proved various existence and uniqueness theorems for differential and integro-differential equations with nonlocal boundary conditions and obtained the necessary optimality conditions in optimal control problems described by them

Theoretical and practical value of the study. The results obtained in the dissertation are theoretical and practical. These results can be used to solve applied problems with nonlocal condition and optimal control problems.The schemes used in this work can be used to study other non-local boundary value problems.

Approbation of work. The results of the dissertation work were reported at the seminar of Baku State University (adv.full member of ANAS, prof M.F. Mehdiyev), Republic Scientific Conference (Sheki 2016), "International scientific conference on theoretical and applied problems of mathematics" (Sumgait, 2017), “8th International Conference on Differential and Functional Equations” (Moscow, 2017), International Scientific Conference "Analysis and Applied Mathematics" (Mersin, Turkey, 2018), International scientific-practical conference (Grozny, 2018).

Personal contribution of the author is in formulation of the goal and choice of research direction. Furthermore, all conclusions and the obtained results and research methods belong personally to the author.

Publications of the author. Publications in editions recommended by HAC under President of the Republic of Azerbaijan – 7 papers, conference materials – 2, abstracts of papers – 3.

Institution where the dissertation work was executed.

The work was performed at the department of "Mathematical Analysis" of Ganja State University.

Structure and volume of the dissertation (in signs, indicating the volume of each structural subsection separately).

The dissertation work consists of introduction, two chapters, results, a list of refernces of 52 names. Total volume of the work

236928 signs (title pages -308 signs, table of contents -4620 signs, introduction -50000 signs, chapter I -120000 signs, chapter II -65000 signs).

THE MAIN CONTENT OF THE DISSERTATION

The work consists of an introduction, two chapters, a conclusion and a list of used literature.

In the introduction rationale of the research work is justified, degree of its elaboration is shown, goal and takes of the research are formulated, scientific novelty is reduced, theoretical and practical value is noted, information on approbation of the work is given.

The first chapter of the dissertation is devoted to finding sufficient conditions that provide the existence and uniqueness of the solution of differential and integro-differential equations with nonlocal condition.

In the first paragraph of the first chapter, the following boundary value problem is considered.

Let's investigate the existence and uniqueness of solution of the following differential equation with nonlocal condition and impulse effect :

$$\dot{x}(t) = f(t, x(t)), \qquad t \in [0,T], t \neq t_i, \qquad i = 1, 2, ..., p. \tag{1}$$

subject to

$$Ax(0) + \int_0^T n(t)x(t)dt = B \tag{2}$$

and

$$\Delta x(t_i) = I_i(x(t_i)), \qquad i = 1, 2, ..., p\ . \tag{3}$$

Here $0 = t_0 < t_1 < ... < t_p < t_{p+1} = T$ are given points, $A \in R^{nxn} -$ is given matrix, $n(t) \in R^{n \times n}$ -is given matrix-function və $\det N \neq 0,$ here $N = A + \int_0^T n(t)dt,$ $I_i : R^n \to R^n$ və $f : [0,T] \times R^n \to R^n$ are given functions, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$ is assumed that here $x(t_i^+) = \lim_{h \to 0^+} x(t_i + h),$ $x(t_i^-) = \lim_{h \to 0^+} x(t_i - h) = x(t_i)$ are left- and right hand limits of $x(t)$ at $t = t_i$ respectively.

Note that, the problem (1)-(3) is a general enough. Several boundary value problems are obtained from (1)-(3) in special case. Below are some definitions and main facts that will be used. Let $C([0,T];R^n)$ be a vector -function $x(t)$ space of dimension n. It is clear that, the space is Banax and here norm is defined as $\|x\| = \max\limits_{[0,T]} |x(t)|$, $|\cdot|$ -is norm on R^n.

Let $PC([0,T],R^n)$- be the following linear space:

$PC([0,T],R^n) = \{x:[0,T] \to R^n; \quad x(t) \in C((t_i,t_{i+1}],R^n)$, here $x(t_i^+)$ and $x(t_i^-)$ $i = 1,2,\ldots,p$ are finite; $x(t_i^-) = x(t_i)\}$.

Obviously, the linear space $PC([0,T],R^n)$ is Banax and here norm is defined as below :

$$\|x\|_{Pc} = \max\{\|x\|_{C((t_i,t_{i+1}])}, i = 0,1,\ldots,p\}$$

Let's give a definition the problem (1) - (3) as follows.

Definition. *Assume that, the function* $x \in PC([0,T]:R^n)$ *satisfies the following conditions:*

a). For every $t \in [0,T]$, $t \neq t_i$, $i = 1,2,\ldots,p$,

$$\dot{x}(t) = f(t,x(t));$$

b)For $t = t_i$ $i = 1,2,\ldots,p$ $0 < t_1 < t_2 < \ldots < t_p < T$

$$\Delta x(t_i^+) = x(t_i^+) - x(t_i) = I_i(x(t_i));$$

c)The function $x(t) \in PC([0,T],R^n)$ *satisfies the condition (2) ;*

So, the function $x(t) \in PC([0,T],R^n)$ *is called a solution of the problem(1)-(3).*

It is proved that, the problem (1)-(3)is equivalent to

$$x(t) = N^{-1}B + \int_0^T K(t,s)\, f(s,x(s))\,ds + \sum_{i=1}^{P} K(t,t_i)\, I_i(x(t_i))$$

Here

$$K(t,\tau)=\begin{cases} N^{-1}(A+\int_0^t n(\tau)d\tau), & 0\le \tau\le t, \\ -N^{-1}\int_t^T n(\tau)d\tau, & t<\tau\le T. \end{cases}.$$

The first main conclusion of this paragraph is as follows.

Theorem 1. *Assume that, the following conditions are satisfied:*

(H1) There exists $M \ge 0$*, for each* $t\in[0,T]$ *and for every* $x,y\in R^n$ *the following inequality is true;*

$$|f(t,x)-f(t,y)|\le M\,|x-y|$$

(H2) There exists constant numbers $l_i \ge 0$*,* $i=1,2,...,p$ *, for each* $x,y\in R^n$ *the following inequality is satisfied;*

$$|I_i(x)-I_i(y)|\le l_i\,|x-y| \ .$$

Besides, if the following inequality is satisfied, then the boundary value problem (1)-(3) has a unique solution;

$$L=S\left(MT+\sum_{k=1}^{p} l_k\right)<1,$$

here S *is defined as* $S=\max_{0\le t,s\le T}\|K(t,s)\|$.

The second main result of this section is based on Schaufer's fixed point theorem and is devoted to the proof of the theorem on the existence of a solution of the boundary value problem (1) - (3).

Theorem 2. *The following conditions are satisfied together with the conditions (H1) and (H2):*

(H3) The function $f:[0,T]\times R^n \to R$ *is continuous and there exists* $N_1 \ge 0$ *- for each* $t\in[0,T]$ *and* $x\in R^n$ *the following inequality is satisfied*

$$|f(x,t)|\le N_1;$$

(H4) The function $I_k : R^n \to R^n$ *is continuous and there exists* $N_2 \ge 0$ *- for each* $x\in R^n$ *the following inequality is satisfied*

$$\max_{k\in\{1,2,\ldots,P\}} |I_k(x)| \le N_2 .$$

Therefore, there is at least one solution of the boundary value problem (1)-(3) on $[0,T]$.

(1) - (3) are given sufficient conditions to ensure the boundary conditions of the solution of the boundary value problem on the right-hand side, on the right-hand side of the differential equation (1) and continuous dependence on the impulse conditions.

The second paragraph of the first chapter deals with the boundary value problem with non-local condition as follows:

$$\dot{x}(t) = f\left(t, x(t), \int_0^t g(t, s, x(s)ds\right), \quad t \in [0,T], \tag{4}$$

$$Ax(0) + \int_0^T n(t)x(t)dt = B. \tag{5}$$

Here it is proved that, the problem (4)-(5) is equivalent to the following integral equation:

$$x(t) = N^{-1}B + \int_0^T K(t,s) f(s, x(s))ds .$$

Theorem 3. *Assume that, the following conditions are satisfied:*

There exist constants $M_1 \ge 0$ *and* $M_2 \ge 0$, *for each* $t \in [0,T]$ *and for arbitrary values of* $(x,y) \in R^{2n}$ *and* $(\bar{x},\bar{y}) \in R^{2n}$ *the following inequalities are true:*

$$| f(t,x,y) - f(t,\bar{x},\bar{y})| \le M_1(|x-\bar{x}| + |y-\bar{y}|),$$

$$|g(t,s,x) - g(t,s,y)| \le M_2|x-y|$$

If the following condition is satisfied,

$$L = S\left(M_1 T\left(1 + \frac{M_2 T}{2}\right)\right) < 1$$

then the nonlocal boundary value problem (4)-(5) has a unique solution

The second main result of this section is devoted to having at least one solution to the boundary value problem under consideration. This result is based on Schaufer's fixed point theorem. In addition, this paragraph finds sufficient conditions for the continuous dependence of the boundary conditions of the solution of the boundary value problem by the right-hand side.

In the third paragraph of the first chapter

$$\dot{x}(t) = f\left(t.x(t), \int_0^t g(t,s,x(s))ds\right), \qquad t \in [0,T], \quad i \neq t_i \qquad i = 1,2,...p \tag{6}$$

for equation (6) the following nonlocal boundary value problem is considered:

$$Ax(0) + \int_0^T n(t)x(t)dt = B. \tag{7}$$

Assume that, the solution of the integro-differential equation (6) satisfies not only the boundary condition (7), but also the impulse condition

$$x(t_i^+) - x(t_i) = I_i(x(t_i)), \quad i = 1,2,..., p \tag{8}$$

Theorem 4. Suppose that, $f \in C([0,T] \times R^n \times R^n; R^n)$, $g \in C([0,T] \times [0,T] \times R^n; R^n)$ *and for* $i = 1,2,\ldots p$ *the conditions* $I_i(x) \in C(R^n; R^n)$ *are true. Necessary and sufficient condition for being a solution of boundary value problem (6)-(8) of the function* $x(t) \in PC([0,T]; R^n)$ *is the function* $x(t) \in PC([0,T]; R^n)$ *to be a solution of the following impulsive integral equation: for* $t \in (t_i, t_{i+1}]$, $\quad i = 0,1,2,\ldots p$,

$$x(t) = N^{-1}B + \int_0^T K(t,s) f\left(s, x(s), \int_0^s g(s,\tau,x(\tau))d\tau\right) + \sum_{i=1}^{p} K(t,t_i) I_i(x(t_i)).$$

The main results of this paragraph are given by the following theorems.

Theorem 5. *Assume that, the following conditions are true:*

(H1) For arbitrary values of $t \in [0,T]$ *and for each* $(x,y) \in R^{2n}$, $(\bar{x},\bar{y}) \in R^{2n}$ *there exist constants* $M_1 \geq 0$ *and* $M_2 \geq 0$ *such that the inequalities*

$$|f(t,x,y) - f(t,\bar{x},\bar{y})| \leq M_1(|x-\bar{x}| + |y-\bar{y}|$$

$$|g(t,s,x) - g(t,s,y)| \leq M_2|x-y|$$

are true.

(H2) There exist $l_i \geq 0, \quad i = 1,2,\ldots p$ *and for every* $x, y \in R^n$

$$|I_i(x) - I_i(y)| \leq l_i|x-y|.$$

If

$$L = S\left(M_1 T\left(1 + \frac{M_2 T}{2}\right) + \sum_{i=1}^{p} l_i\right) < 1$$

Then the boundary value problem (6)-(8) has a unique solution, here S is defined as $S = \max_{0 \leq t,s \leq T} \|K(t,s)\|$.

This section also proves the existence theorem based on Schaufer's fixed point theorem and the theorems on the continuous dependence of the boundary conditions of the solution by the right-hand side.

In the fourth paragraph of the first chapter sufficient conditions will be found for the existence and uniqueness of the solution to the system of integro-differential equations

$$\dot{x}(t) = f(t,x(t)) + \int_0^t g(t,s,x(s))ds, \qquad t \in [0,T] \tag{9}$$

with three-point boundary conditions

$$Ax(0) + Bx(t_1) + Cx(T) = \alpha. \tag{10}$$

Here $A, B, C \in R^{n\times n}$ are given matrices. $\alpha \in R^n$ is given vector of dimension n . Suppose that, $\det N \neq 0$, here $N = A + B + C$, $f : [0,T] \times R^n \to R^n$ and $g : [0,T] \times [0,T] \times R^n \to R^n$ are given functions , t_1 satisfies $0 < t_1 < T$.

Here too, we will denote the space $C([0,T];R^n)$ as continuous functions on $[0,T]$ and apparently, this space is Banax space.

Theorem 6. Necessary and sufficient condition for being a solution of the boundary value problem (9) - (10), the following integral equation must be

$$x(t)=N^{-1}\alpha+\int_0^T G(t,s)\left[f(s,x(s))+\int_0^s g(s,\tau,x(\tau))d\tau\right]ds,$$

here

$$G(t,s)=\begin{cases}G_1(t,s), & t\in[0,t_1]\\ G_2(t_1 s), & t\in[t_1,T]\end{cases}.$$

The functions $G_1(t,s)$ and $G_2(t,s)$

$$G_1(t,s)=\begin{cases}N^{-1}A, & s\in[0,t],\\ -N^{-1}(B+C), & s\in(t,t_1],\\ -N^{-1}C, & s\in(t_1,T],\end{cases}$$

$$G_2(t,s)=\begin{cases}N^{-1}A, & t\in[0,t_1],\\ N^{-1}(A+B), & t\in[t_1,t],\\ -N^{-1}C, & t\in[t,T].\end{cases}$$

are determined by means of equations.

Assume that, the following conditions are true:

(H1) There exists continuous function $l(t)\geq 0$, for arbitrary $t\in[0,T]$ and for each $x,y\in R^n$

$$|f(t,x)-f(t,y)|\leq l(t)|x-y|$$

the inequality is satisfied.

(H2) There exists continuous function $m(t)\geq 0$, for arbitrary $t\in[0,T]$ and for every $x,y\in R^n$

$$\left|\int_0^t g(t,s,x)ds - \int_0^t g(t,s,y)ds\right| \le m(t)|x-y|$$

the inequality is satisfied.

Theorem 7. *Assume that, the conditions (H1) and (H2) are true and*

$$L = G_{\max} T\left[l + \frac{mT}{2}\right] < 1$$

the inequality is satisfied. Then the boundary value problem (9)-(10) has a unique solution on $[0,T]$ *,here* $l = \max_{[0,T]} l(t),\quad m = \max_{[0,T]} m(t), G_{\max} = \max_{[0,T]\times[0,T]} |G(t,s)|$.

Similarly, the second main result of this paragraph, based on Schaefer's fixed point theorem is given.

The second chapter of the dissertation is devoted to the study of optimal control problems, described by boundary value problems with non-local condition.

The first paragraph of the second chapter considers the optimal control problem as follows.

Here, the optimal control problem is described by the boundary value problem given by a system of integro-differential equations with impulse effect and non-local condition.

$$\frac{dx}{dt} = f(t, x(t), u(t)) + \int_0^t g(t,\tau,x(\tau),u(\tau))d\tau, \quad 0 \le t \le T,\ t \ne t_i, \tag{11}$$

$$x(0) + Bx(T) = C, \tag{12}$$

$$\Delta x(t_i) = I_i(x(t_i), v_i), \quad i = 1,2,\dots,p,\ \ 0 < t_1 < t_2 < \dots < t_p < T, \tag{13}$$

$$(u(\cdot),[v]) \in U \times \Pi^p = \{u(t) \in L_2^r[0,T] : u(t) \in V, n.в.t \in [0,T], v_i \in \Pi\}, \tag{14}$$

here $x(t) \in R^n$, the $f(t,x,u)$ is continuous function of dimension n, $B \in R^{n\times n}, C \in R^{n\times 1} -$ are given constant matrices, $\Delta x(t_i) = x(t_i^+) - x(t_i^-)$, $I_i(x,v) -$ are given function of dimension n, $(u,[v])$-are control parameters, $V \in R^r$ və $\Pi \in R^m$- are closed, convex, bounded sets .

On the solution set of the boundary value problem (11) - (14) is required minimization of the following functionality:

$$J(u,[v]) = \Phi(x(0), x(T)) \ . \tag{15}$$

When solving the boundary value problem (11) -(14) for each control parameters $(u(\cdot),[v]) \in U \times \Pi^p$ on $[0,T]$, $t \neq t_i$ we assume that, there exist uniformly continuous vector functions $x(t):[0,T] \to R^n$ and at the points of $t = t_i$ $i = 1,2,\dots,p$ these functions are continuous by left-hand side and there exist finite $x(t_i^+)$ right-hand limits.

Here it is assumed that, the following conditions are satisfied.

1). $\|B\| < 1$.

2). $f:[0,T]\times R^n \times R^r \to R^n$, $g:[0,T]\times[0,T]R^n \times R^r \to R^n$ and the functions $I_i : R^n \times R^m \to R^n$, $i = 1,2,\dots,p$ are continuous and there exist $K > 0, G > 0$, $L_i > 0$ $i = 1,2,\dots,p$,

$$|f(t,x,u) - f(t,y,u)| \le K|x-y|, \quad t\in[0,T], \ x,y \in R^n;$$

$$|g(t,\tau,x,u) - g(t,\tau,y,u)| \le G|x-y|, \quad t,\tau\in[0,T], \ x,y \in R^n$$

;

$$|I_i(x,v) - I_i(y,v)| \le L_i|x-y|, \quad x,y \in R^n;$$

the above inequalities are satisfied.

3).

$$L = (1-\|B\|)^{-1}[KT + \frac{GT^2}{2} + \sum_{i=1}^{p} L_i] < 1 .$$

Theorem 8. Assume that, *condition 1) is satisfied.*. Necessary and sufficient condition for being a solution of the boundary value problem (11) -(13), $x(\cdot)\in PC\left([0,T],R^n\right)$ *must be a solution of* the following integral equation

$$x(t)=(E+B)^{-1}C+$$
$$+\int_0^T K(t,\tau)\left\{f(\tau,x(\tau),u(\tau))+\int_0^\tau g(\tau,s,x(s),u(s))ds\right\}d\tau+$$
$$+\sum_{i=1}^{p}K(t,t_i)I_i(x(t_i),v_i) \tag{16}$$

here

$$K(t,\tau)=\begin{cases}(E+B)^{-1}, & 0\le\tau\le t\\ -(E+B)^{-1}B, & t\le\tau\le T\end{cases}.$$

Theorem 9. *Suppose that, the conditions (1.-3) are satisfied.Then for every* $C\in R^n$ *and* $\left(u(\cdot),[v]\right)\in U\times\Pi^p$ *the boundary value problem (11)-(13) has a unique solution and the solution is a solution of the following integral equation :*

$$x(t)=(E+B)^{-1}C+$$
$$+\int_0^T K(t,\tau)\{f(\tau,x(\tau),u(\tau))+\int_0^\tau g(\tau,s,x(s),u(s))ds\}d\tau+ \tag{17}$$
$$+\sum_{i=1}^{P}K(t,t_i)I_i(x(t_i),v_i).$$

To calculate the gradient of a function in optimal control problem, let us include the following conditions.

4). The derivative of functions $f(t,x,u)$ and $g(t,s,x,u)$ with repect to u is bounded , that is

$$|f_u(t,x,u)\bar{u}|\le K_1|\bar{u}|$$
$$|g_u(t,x,u)\bar{u}|\le K_1^g|\bar{u}|.$$

5). The derivative of functions $f(t,x,u)$ and $g(t,s,x,u)$ with repect to x and u satisfies Lipschitz condition, that is, the following inequalitie

$$\left| f(t,x+\bar{x},u+\bar{u}) - f(t,x,u) - f_x(t,x,u)\bar{x} - f_u(t,x,u)\bar{u} \right| \leq$$

$$\leq K_2 |\bar{x}|^2 + K_3 |\bar{u}|^2,$$

$$\left| g(t,s,x+\bar{x},u+\bar{u}) - g(t,s,x,u) - g_x(t,s,x,u)\bar{x} - g_u(t,s,x,u)\bar{u} \right| \leq$$

$$\leq K_2^g |\bar{x}|^2 + K_3^g |\bar{u}|^2$$

are true.

6). The derivative of function $I_i(x,v)$, $i = 0,1,...,p$ with repect to v is bounded, that is, the following inequality

$$\left| I_{iv}(x,v)\bar{v} \right| \leq L_i^{(1)} |\bar{v}|$$

is true.

7). The derivative of functions $I_i(x,v)$, $i = 0,1,...,p$ with repect to x and v satisfies Lipschitz condition, that is, the following inequality

$$\left| I_i(x+\bar{x},v+\bar{v}) - I_i(x,v) - I_{ix}(x,v)\bar{x} - I_{iv}(x,v) \right| \leq L_i^{(2)} |\bar{x}|^2 + L_i^{(3)} |v|^2$$

is satisfied.

8). The first partial derivatives of function $\Phi(x,y)$ are bounded and these derivatives satisfy Lipschitz condition, that is,

$$\left| \Phi_x(x,y) \right| \leq K_4; \left| \Phi_y(x,y) \right| \leq K_5.$$

$$\left| \Phi(x+\bar{x},y+\bar{y}) - \Phi(x,y) - \langle \Phi_x(x,y),\bar{x} \rangle - \langle \Phi_y(x,y),\bar{y} \rangle \right| \leq K_6 |\bar{x}|^2 + K_7 |\bar{y}|^2.$$

Lemma 1. *Assume that, the conditions (1).-4)are satisfied.,* $(u(t),[v],x(t))$ *and* $(u(t)+\bar{u}(t),[v+\bar{v}],x(t)+\bar{x}(t))$ *are different solutions of optimal control problem (11)-(14).Then*

$$\left|\bar{x}(t)\right| \le C_1\left(\left\|\bar{u}\right\| + \left\|[\bar{v}]\right\|\right) \tag{18}$$

the evaluation is correct. Here

$$C_1 = (1-L)^{-1}\left(1-\left\|B\right\|\right)^{-1} \max\left[\left(K_1\sqrt{T} + K_1^g T^{\frac{3}{2}}\right), \max L_i^{(1)}\sqrt{p}\right].$$

Lemma 2. *Assume that, the conditions (1)-6) are satisfied and* $(\bar{u}(t),[\bar{v}],\bar{x}(t))$ *are the functions defined in lemma 1,. The function* $z(t)$ *is the solution of the equation with variations. Then*

$$\left|\bar{x}(t) - z(t)\right| \le C_2\left(\left\|\bar{u}\right\|^2 + \left\|[v]\right\|^2\right), \quad 0 \le t \le T\,, \tag{19}$$

here

$$C_2 = \max\left\{(1-L)^{-1}\left(1-\left\|B\right\|\right)^{-1}\left(2K_2C_1^2 + K_3 + 2K_2^g + K_3^g T^{\frac{3}{2}}\right),\right.$$

$$\left.(1-L)^{-1}\left(1-\left\|B\right\|\right)^{-1}\left(2K_2 + G^2 + 2K_2^g T^2 + p\max_{1\le i\le p} L_i^{(3)}\right)\right\}$$

Theorem 10. *Suppose that, the conditions (1)-8) are satisfied and besides*

$$det\left(E + \frac{\partial I_i\left(x(t_i),v_i\right)}{\partial x}\right) \ne 0, \quad i = 1,2,\ldots p.$$

Then functional (15) is differentiable within the conditions (11)-(14) and its gradient

$$J'(u_1[v]) = \left(\frac{\partial H(t,x,u,\psi)}{\partial u}; \frac{\partial h_i\left(x_i,v_i\right)\psi(t_i)}{\partial x_i}\right) \in L_2^r([0,\tau]) \times R^n, \tag{20}$$

here

$$H(t,x,u,\psi) = \langle\psi, f(t,x,u)\rangle + \int_t^T \langle\psi, g(t,\tau,x,u)\rangle d\tau,$$

$$h_i\left(x_i,u_i,\psi(t_i)\right) = <\psi(t_i), I_i\left(x_i,v_i\right)>$$

designated as above, and the function $\psi(t)$ *is the solution of the boundary value problem for the differential-difference equation as follows:*

$$\frac{d\psi}{dt} = -\frac{\partial H^{'}(t,x,u,\psi)}{\partial x}, \ t \pm t_i, \quad i = 1,2,\ldots p, \tag{21}$$

$$\Delta\psi(t_i) = -\frac{\partial I_i^{'}(x_i, v_1)}{\partial x}\left(\frac{\partial I^{'}(x_i v_i)}{\partial x} + E\right)\psi(t_i) \quad i = 1,2,\ldots p \tag{22}$$

$$\left(E + B^{'}\right)^{\psi}(T) + B^{'}\left(E + B^{'}\right)^{-1}\psi(0) = \tag{23}$$

$$= B^{'}\left(E + B^{'}\right)^{-1}\frac{\Phi(x(0), x(\tau))}{\partial x(0)} - (E + B)^{-1}\frac{\partial\Phi}{\partial x(\tau)}$$

Theorem 11. *Suppose that, conditions of theorem 10 are satisfied. Then the necessary condition for optimization of the control* $(u_*, [v]_*) \in U \times \Pi^p$ *in optimal control problem (21)-(25) for every* $(u,[v]) \in U \times \Pi^p$ *is satisfying of the following inequality*

$$\int_0^T \langle H_u(t, x_*(t), u_*(t), \psi_*(t)), u(t) - u_*(t)\rangle dt + \sum_{i=1}^{p}\langle h_{iv_i}(x_{i*}, v_{i*}), v_i - v_{i*}\rangle \geq 0$$

, here $x_*(t) = x(t; u_*, [v]_*), \psi_*(t) = \psi(t; u_*, [v]_*)$.

In the second paragraph of the second chapter, it is assumed that the controlled process is described by a system of differential equations as follows.

It is assumed that for the differential eqauation (44)

$$\dot{x}(t) = f(t,x,u) \tag{24}$$

the following condition

$$Ax(0) + \int_0^T m(t)x(t)dt = C. \tag{25}$$

is satisfed. Admissible controls get their values from a nonempty bounded set U, that is

$$u(t) \in U \subset R^r, \ t \in [0,T].$$

It is requirred that, there exists the control $u(t) \in U \subset R^r$, $t \in [0,T]$ is to be found that, a respective solution to the boundary value problem (24), (25) is to give minimizing to functional:

$$J(u) = \varphi(x(0), x(T)) + \int_0^T F(t,x,u)dt \tag{26}$$

Here it is assumed that, the following conditions are satisfied:

(A1) $\det N \neq 0,$ where $N = A + \int_0^T m(t)dt.$

(A2) the function $f : [0,T] \times R^n \times R^n \to R^n$ is continuous and there exists $K \geq 0$ such that,

$$\left| f(t,x,u) - f(t,y,u) \right| \leq K|x-y|,\ t \in [0,T],\ x,y \in R^n, u \in U \ .$$

(A3) $L = KTM < 1,$

here $M = \max_{0 \leq t,s \leq T} \|M(t,s)\|,$ the matrix function $M(t,s) -$ is defined by means of the following equality

$$M(t,s) = \begin{cases} N^{-1}\left(A + \int_0^t m(s)ds \right), & s < t, \\ -N^{-1}\int_t^T m(s)ds, & t \leq s. \end{cases} \tag{27}$$

Theorem 12. *Suppose that, A1) is satisfied. The necessary and sufficient condition for being a uniformly continuous solution of boundary value problem (24)-(25) of the function* $x(\cdot) \in C([0,T], R^n)$ *is that the function must be a solution of the following integral equation*

$$x(t) = N^{-1}C + \int_0^T M(t,s) f(s, x(s), u(s))ds,$$

Here matrix function $M(t,s)$ *is defined by means of equality (27)*

Here to increase the functionality (26) with the using of standard operations, is obtained the following formula:

$$\Delta J(u) = -\int_{\theta}^{\theta+\varepsilon} \Delta_{\upsilon} H(t,\psi,x,u)dt + o(\varepsilon)\,. \tag{28}$$

From the formula (28) is gotten the maximum principle of Pontryagin.

Theorem 13. (Maximum prinsipi) *Assume that, the process(* $u^0(t), x^0(t,u^0)$ *) is optimal in optimal control problem (24) - (26)* $\psi^0(t)$ -

$$\dot{\psi}(t) = -\frac{\partial H(t,\psi,x,u)}{\partial x} - m'(t)\lambda, \qquad t \in [0,T],$$

$$\psi(0) = A'\lambda + \frac{\partial \varphi}{\partial x(0)}, \quad \psi(T) = -\frac{\partial \varphi}{\partial x(T)},$$

is a solution of conjugate problem.Then as if for all $t \in [0,T]$ *the following equality is satisfied:*

$$\max_{\upsilon \in U} H(t,\psi^0(t), x^0(t), \upsilon) = H(t,\psi^0(t), x^0(t), u^0(t)). \tag{29}$$

In the third paragraph of the third chapter, it is assumed that the state of the controlled object is described by the following integro-differential equations:

$$\dot{x}(t) = f(t, x(t), u(t)) + \int_0^t k(t,\tau, x(\tau), u(\tau))d\tau\,, \qquad t \in [0,T], \tag{30}$$

Equation (30) is given by the two-point boundary condition as follows

$$Ax(0) + Bx(T) = C\,. \tag{31}$$

Here it is assumed that, $f(t,x,u)$ and $k(t,\tau,x,u)$ are given functions of dimensional and respectively $[0,T]\times R^n \times R^r$ and are continuous on set of $[0,T]\times[0,T]\times R^n \times R^r$ and have continuous partial derivatives up to the second order with respect to variables (x,u). $A, B \in R^{n\times n}$ and $C \in R^{n\times 1}$ are given matrices of dimensional, $[0,T]$ is a given segment.. $u = u(t)$ r- are dimensional

piece-wise continuous vector functions (has type I breakpoints at a finite number of points).
These functions get their values from a given non-empty bounded and compact set $U \subset R^r$, that is

$$u(t) \in U \subset R^r, \quad t \in [0,T]. \tag{32}$$

The control functions that satisfy condition (32) are called admissible control.
For each of these admissible controllers, let us define the functional defined in the set of solutions of the problem (30), (31):

$$J(u) = \varphi(x(0), x(T)). \tag{33}$$

Here $\varphi(x, y)$ is a twice differentiated scalar function defined in the set of $R^n \times R^n$.
Suppose that, the following conditions are satisfied:
I). $\det (A+B) \neq 0$
II). $f : [0,T] \times R^n \times R^r \to R^n$ are $k : [0,T] \times [0,T] \times R^n \times R^r \to R^n$ continuous functions and there exist $K \geq 0$ and $L \geq 0$ such that

$$| f(t, x, u) - f(t, y, u)| \leq K|x - y|, t \in [0,T], (x, y) \in R^{2n}, u \in R^r,$$

$$|k(t, \tau, x, u) - k(t, \tau, y, u)| \leq L|x - y|.$$

III) $ST(K + LT) < 1$, here

$$S = \max\left\{\left\|(A+B)^{-1} A\right\|, \left\|(A+B)^{-1} B\right\|\right\}.$$

Theorem 14. *Assume that, the conditions I) – III) are satisfied. Then for any* $C \in R^n$ *and for any control the boundary value problem (30) – (31) has a unique solution.*
Pontryagin's maximum principle for the optimal control problem under consideration has been proved.

Theorem 15. (Maximum principle) *Assume that, the pairs* $(u^0(t), x^0(t, u^0))$ *are optimal process in optimal control problem (30)-(33),* $\psi^0 = \psi^0(t)$ *throughout optimal process*

$$\dot{\psi} = -H_x(t, x(t), u(t), \psi(t)),$$

$$B(A+B)^{-1}\psi(0)+A(A+B)^{-1}\psi(T)=$$
$$=B(A+B)^{-1}\frac{\partial\varphi}{\partial x(0)}-A(A+B)^{-1}\frac{\partial\varphi}{\partial x(T)}$$

is a solution of conjugate boundary value problem. Then for any $v\in U$ *the equality (29) is true.*

CONCLUSION

The following crucial results were obtained in the dissertation.

1. Theorems on the existence and uniqueness of the solution of a system of nonlinear differential equations with nonlocal condition and impulse effect have been proved.
2. Sufficient conditions have been found for the continuous dependence of the solutions of a system of nonlinear differential equations with nonlocal condition and impulse effect on the boundary conditions, on the right-hand side of the system of differential equations, and on impulse effects.
3. Theorems on the existence and uniqueness of the solution of a system of nonlinear integro- differential equations with nonlocal condition have been proved.
4. Theorems on the existence and uniqueness of the solution of a system of nonlinear integro-differential equations with nonlocal condition and impulse effect have been proved.
5. The theorems on the existence and uniqueness of the solution of a system of nonlinear integro-differential equations with three-point boundary condition have been proved.
6. Necessary conditions for optimality in the form of variational inequality and maximum principle in the problems of optimal control for a system of integro-differential equations with two-point boundary condition have been found.
7. The maximum principle has been proved in optimal control problem described by system of differential equations with integral condition.

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